Strings inside walls in $\mathcal{N} = 1$ super Yang–Mills

2009 J. Phys. A: Math. Theor. 42 195404

(http://iopscience.iop.org/1751-8121/42/19/195404)

IOP PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND THEORETICAL

J. Phys. A: Math. Theor. **42** (2009) 195404 (22pp) doi:10.1088/1751-8113/42/19/195404

Strings inside walls in $\mathcal{N} = 1$ super Yang–Mills

S Bolognesi

William I. Fine Theoretical Physics Institute, University of Minnesota, 116 Church St. S.E., Minneapolis, MN 55455, USA

E-mail: bolognesi@physics.umn.edu

Received 31 October 2008, in final form 19 March 2009
Published 22 April 2009
Online at stacks.iop.org/JPhysA/42/195404

Abstract
We conjecture the existence of strings bounded inside walls in $\mathrm{SU}(n)\mathcal{N} = 1$ super Yang–Mills theory. These strings carry a $\mathbb{Z}_{[k,n]}$ quantum number, where $[k, n]$ is the greatest common divisor between k, the charge of the wall and n. We provide field-theoretical arguments and string-theoretical evidences, both from MQCD and from gauge-gravity correspondence. We interpret this result from the point of view of the low-energy effective action living on the k-wall.

PACS numbers: 11.15, 11.27.+d, 12.60.Jv

(Some figures in this article are in colour only in the electronic version)

1. Introduction

In this paper, we want to show a particular relation between confining strings and domain walls in $\mathrm{SU}(n)\mathcal{N} = 1$ super Yang–Mills in four dimensions. This theory has no matter fields charged under the center $\mathbb{Z}_n$ of the gauge group. As any theory in this class, confinement, when it happens, is associated with the existence of the so-called k-strings. An external probe source with k color indices (or n-ality k) is confined by the k-string. This feature is in particular also shared by the ordinary $\mathrm{SU}(n)$ Yang–Mills theory. Much work, on both the lattice and the theoretical sides, has been done on this subject, in particular on the study of the tension dependence versus k and n.[1]

Besides k-strings, the theory under consideration has other topological defects with a $\mathbb{Z}_n$ charge. These are the domain walls that separate two of the n discrete vacua of the theory. The aim of this work is to show that $\mathbb{Z}_{[k,n]}$ confining strings live inside a k-wall of $\mathrm{SU}(n)\mathcal{N} = 1$ SYM. From now on, we shall indicate as $[k, n]$ the maximum common divisor between k and n.

We have a series of arguments to support our claim. We start in section 2 with a heuristic argument. In section 3, we give a more substantial argument from the MQCD realization of

[1] See [34–37] for generic references.

the theory. In section 4, we discuss the gauge-gravity setup and how to understand our result. In section 5 we provide a field theoretical proof, using $\mathcal{N} = 2$ SYM softly broken. Here, we will discuss the important analogy with the recent work [14]. In section 6 we discuss the peeling issue. We then move to the low-energy effective theory in $2 + 1$ dimensions in section 7, and see why this effect cannot be seen from the $2 + 1$ effective action. At the end, we summarize our results in section 8. The appendix contains a discussion about the string–wall junction.

2. Heuristic argument

Let us start with some basic facts about $\mathcal{N} = 1$ SU(n) super Yang–Mills. The theory has a $U(1)_R$ axial symmetry broken by an anomaly to $\mathbb{Z}_{2n}$. This remnant symmetry is further broken to $\mathbb{Z}_2$ by the gluino condensate $\langle\lambda\lambda\rangle \propto n\, \mathrm{e}^{\mathrm{i}2\pi k/n}\Lambda^3$; the theory thus possesses n degenerate vacua labeled by a $\mathbb{Z}_n$ number. Two distinct vacua, let us say h-vacuum and $(h + k)$-vacuum, can be separated by a domain wall which we shall denote as a k-wall. These walls are $1/2$ BPS saturated and their tension is equal to the modulus of the difference of the superpotentials between the two vacua [1]. Each of these vacua is in a massive phase where probe quarks are confined. According to 't Hooft's classification of massive phases, we must specify the $\mathbb{Z}_n^{\mathrm{ele}} \times \mathbb{Z}_n^{\mathrm{mag}}$ charges of the particles that condense and are responsible for confinement. In the h-vacuum of $\mathcal{N} = 1$ SYM, confinement is due to the condensation of an $(h, 1)$ particle.

In $\mathcal{N} = 1$ SYM there are thus two interesting extended objects, both labeled by a $\mathbb{Z}_n$ number: domain walls and confining strings. We shall now show that a bound state between these two objects exists. A $\mathbb{Z}_{[k,n]}$ string can live inside a k-wall.

Something is already known about the relation between strings and walls. For example, as first noted in [2], a 1-string can end on a 1-wall. A simple field theoretical argument goes as follows. A 1-string can terminate on a probe quark or on any object with charges $(1, 0)$. A particle with these charges can be created if the two condensates living on the two sides of the wall can form a bound state. If, for simplicity, we consider the 1-wall separating the 0 and the 1 vacua, the two condensates are, respectively, $(0, 1)$ and $(1, 1)$. A bound state $-(0, 1) + (1, 1)$, an anti-monopole plus a dyon, would thus have the charge of a fundamental quark and thus be a good ending point for a 1-string.

We now want to use the same argument and push it further, considering a generic h-string perpendicular to a generic k-wall. Can this string end, or not, on the domain wall? It would be possible if there were a composite of the two condensates on which the h-string could end. An h-string can end by definition on h quarks or on any bound state with charge $(h, 0)$. To be general, we consider the k-wall that interpolates between the p and the $p + k$ vacua, where the two condensates responsible for confinement are, respectively, $(p, 1)$ and $(p + k, 1)$. We thus want to solve the following equation for a and b:

$$(h, 0) = a(p, 1) + b(p + k, 1). \tag{2.1}$$

The equation is defined in the ring $\mathbb{Z}_n^{\mathrm{ele}} \times \mathbb{Z}_n^{\mathrm{mag}}$. If the equation is solvable, the string can end on the wall; if not, the string cannot end on the wall and is forced to continue further (see figure 1). From the monopole charges, the second in $(.,.)$, we get $a = -b$, and from the electric charges we get

$$h = bk \quad \text{modulo } n. \tag{2.2}$$

A simple theorem from arithmetic tells us that this equation is solvable if and only if h is a multiple of the greatest common divisor between k and n.

We have thus the following scenario. When a k-wall interpolates between a p and a $p + k$ vacua, the group of $\mathbb{Z}_n$ strings is divided into two categories. The subgroup $\mathbb{Z}_{n/[k,n]}$ of strings

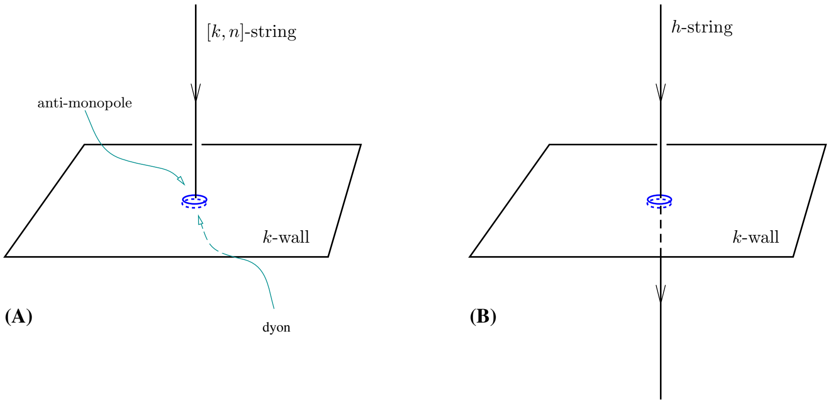

Figure 1. (*A*) An h-string can end on a k-wall only if h is a multiple of $[k, n]$. (*B*) If h is not divisible by $[k, n]$, the h-string cannot terminate on the wall and is forced to continue on the opposite vacuum.

multiple of $[k, n]$, can terminate on the wall. The quotient group $\mathbb{Z}_{[k,n]}$ is instead topologically stable. Another way to say this is that a string crossing the k-wall can change its n-ality but cannot change its $[k, n]$-ality.

The natural question is now what happens if the string, instead of being perpendicular to the wall, is *parallel* to the wall. A string in general feels an attractive force toward a parallel domain wall. For example, a 1-string is attracted toward a 1-wall and then dissolved. We thus conclude that any h-string parallel to a k-wall feels an attractive force. When the h-string arrives on the domain wall, its fate is determined by the divisibility of h. If h is a multiple of $[k, n]$, the string is dissolved inside the wall and disappears. If not, the string survives as a $\mathbb{Z}_{[k,n]}$ string bounded inside the domain wall.

The other two deformations, described in figure 2, reveals the existence of this bound state of confining strings inside the domain wall.

3. MQCD

Let us now consider the MQCD realization of $\mathcal{N} = 1$ SYM. For a detailed description, we refer to the original papers [2, 4]; in the following, we just present what is needed to understand our result in the MQCD context.

We have M-theory compactified on an S^1 circle. The coordinates are $x^0, \ldots, x^9$ and x^{10}, the M-theory circle, of period 2π. We define the complex coordinates $v = x^4 + \mathrm{i}x^5$, $w = x^7 + \mathrm{i}x^8$ and $t = \mathrm{e}^{-(x^6+\mathrm{i}x^{10})}$. The gauge theory of interest is the low-energy limit of the effective action living on the world volume of an M5 brane. The 5 brane is extended on the four-dimensional spacetime $x^0, \ldots, x^3$ times a non-compact Riemann surface Σ_p. The Riemann surface is defined by the following equations:

$$v^n = t, \qquad w = \zeta_p v^{-1}, \tag{3.1}$$

where ζ_p is the root of unity $\mathrm{e}^{\mathrm{i}2\pi p/n}$. The Riemann surface, Σ_p, encodes certain information about the gauge theory. Roughly, it can be seen as two infinite planes, the v plane at $x^6 \to -\infty$ and the w plane at $x^6 \to +\infty$, connected by a tube that winds n times around the circle of M-theory. There are n ways to connect the two planes depending on the choice of the root of

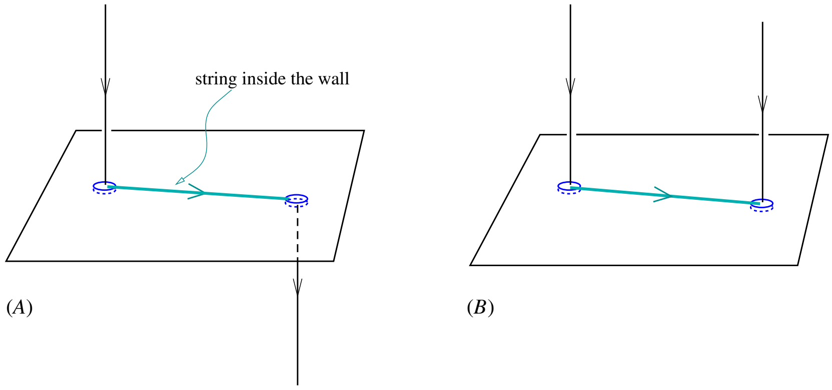

Figure 2. (A) Figure 1(B) is modified so that the strings in the two vacua end on different points on the domain wall. For consistency there should be a string, living inside the domain wall, that connects the two ending points. (B) An h-string and a $-h$-string and on the k-wall on two different points. If h is not a multiple of $[k, n]$, a string inside the domain wall must be created in order to connect the ending points.

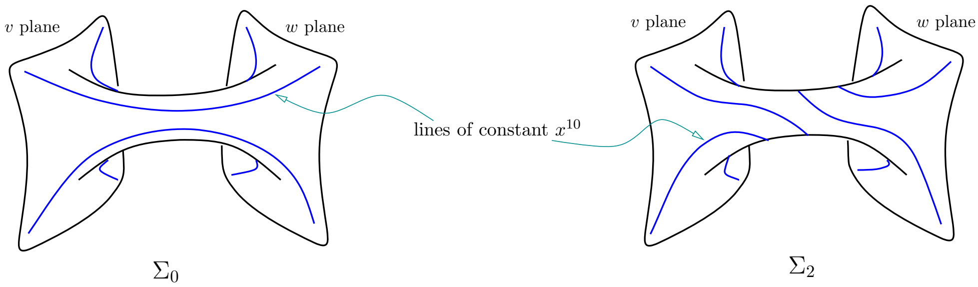

Figure 3. We provide two examples of the Riemann surface for $n = 4$ (we choose this since the 2-wall in SU(4)$\mathcal{N} = 1$ is the first non-trivial case in which our phenomenon happens). In the figure, we have Σ_0 and Σ_2 corresponding respectively to the vacua 0 and 2. The lines represent the phase in the M-theory circle. The domain wall between these two vacua is represented in figure 4. Note that the v and w planes, despite what it seems from the figure, are not parallel, but orthogonal.

unity ζ_p. These correspond to the n discrete vacua of the gauge theory (see figure 3 for an example).

Another feature of the gauge theory, nicely visible in this M-theoretical framework, is the presence of confining $\mathbb{Z}_n$ strings. They correspond to M2 branes with 1 + 1 dimensions extended in spacetime, and the other spatial dimension extended on a finite segment whose end points lie on the surface Σ_p. We call, for convenience, Y the six-dimensional manifold $R^5 \times S^1 = x^{4,\ldots,8}, x^{10}$. The Riemann surface, Σ_p, can thus be thought of as embedded in this six-dimensional space $\Sigma_p \subset Y$. Finite segments in Y, whose end points are forced to be on Σ_p, correspond to elements of the relative homology group $H_1(Y/\Sigma_p, \mathbb{Z})$. From the exact sequence, we know how to express this relative homology group as a function of the homology groups of the Y manifold and of the Σ_p surface:

$$H_1(Y/\Sigma_p, \mathbb{Z}) \cong \frac{H_1(Y, \mathbb{Z})}{\mathrm{i}(H_1(\Sigma_p, \mathbb{Z}))}. \tag{3.2}$$

$H_1(Y, \mathbb{Z})$, equal to $\mathbb{Z}$, simply counts the windings around the M-theory circle. $H_1(\Sigma_p, \mathbb{Z})$ is also equal to $\mathbb{Z}$, and the unit element consists of a winding around the tube connecting the v plane with the w plane. After the immersion $i()$ into Y, this unit element corresponds to n windings around the M-theory circle. We thus conclude that

$$H_1(Y/\Sigma_p, \mathbb{Z}) = Z_n. \tag{3.3}$$

The fate of strings in the presence of a domain wall is again determined by a relative homology group. But now we have to extend the spaces in order to take into account the x^3 direction along which the wall interpolates between the two vacua. We define the space $\widetilde{Y}$ as the seven-dimensional manifold $x^3 \times Y$. The M5 brane extends on a three-dimensional subspace $S_k \in \widetilde{Y}$. S_k has the following features. At $x^3 \to -\infty$ it approaches the surface Σ_p while at $x^3 \to +\infty$ it approaches the surface Σ_{p+k}. S_k is thus a domain wall that interpolates between the p vacuum and the $p + k$ vacuum. Strings in the domain wall background are represented by elements of the relative homology group $H_1(\widetilde{Y}/S_k, \mathbb{Z})$. Again, from the exact sequence we can derive the formula

$$H_1(\widetilde{Y}/S_k, \mathbb{Z}) \cong \frac{H_1(\widetilde{Y}, \mathbb{Z})}{\mathrm{i}(H_1(S_k, \mathbb{Z}))}. \tag{3.4}$$

$H_1(\widetilde{Y}, \mathbb{Z})$ is equal to $\mathbb{Z}$ and still winds around the M-theory circle. The task is now to evaluate $H_1(S_k, \mathbb{Z})$ and embed it into the space $\widetilde{Y}$. We start with the 1-wall already considered in [2]. S_1 has two non-trivial cycles, $H_1(S_1, \mathbb{Z}) = \mathbb{Z}^2$. The first one is the same as the Riemann surfaces Σ's. It is a circle that winds around the tube connecting the v plane and the w plane. After embedding, it corresponds to n times the unit cycle of $H_1(\widetilde{Y}, \mathbb{Z})$. The other non-trivial cycle is peculiar to the domain wall and is constructed as follows. We divide the cycle into five distinct open segments and then we connect them to form a closed circle. The first four pieces make a very large loop in the x^3, x^6 plane, constant in the M-theory circle x^10 and not closed in the v plane. The fifth and last piece closes the curve and winds once around x^{10}. All the following movements are done staying inside the manifold S. The construction, which can be seen in figure 4, goes as follows. We start at $x^3 \to -\infty$ in the Σ_p vacuum and at $x^6 \to +\infty$ in the w plane. We then move toward $x^6 \to -\infty$ into the v plane on a line with a fixed x^{10} coordinate. This is the first movement. The second step is to move toward $x^3 + \infty$, in the Σ_{p+1} region, keeping fixed x^6, v and the x^{10} phase. The third stage is to move back into the $x^6 \to +\infty$ region keeping x^3 fixed and the phase x^{10} constant. In the fourth passage we return to the vacuum Σ_p closing the circle in the x^3, x^6 plane. The coordinate w has been rotated by a phase $\mathrm{e}^{\frac{\mathrm{i}2\pi}{n}}$. To close the circle we make the fifth and last movement, keeping x^3 and x^6 both fixed and rotating around the w plane. This inevitably makes a winding in the x^{10} circle. These five steps connected together form a closed circle, lying entirely in the manifold S_1, that rotates once in the M-theory circle. This is enough to conclude that $H_1(\widetilde{Y}/S_1, \mathbb{Z}) = 1$, which implies that every string can be 'unwinded' in the domain wall background. The same argument can be repeated for a generic k-wall S_k interpolating between the vacua Σ_p and Σ_{p+k}. S_k again has two non-trivial cycles. The first one is the same as the Riemann surfaces Σ's and rotates n times around the M circle. The second is constructed in the same way as before with the only difference being that it rotates k times around x^{10} instead of only once (see figure 4). Thus, the relative homology group in the k-wall background is

$$H_1(\widetilde{Y}/S_k, \mathbb{Z}) = \mathbb{Z}_{[k,n]}. \tag{3.5}$$

It thus implies that $\mathbb{Z}_{[k,n]}$ strings are stable in the k-wall background. If perpendicular to the wall, the $\mathbb{Z}_{[k,n]}$ strings can cross it without changing their $[k, n]$ quantum number. If parallel to the wall, they shall set in the most energetically favorable x^3 position. For a detailed

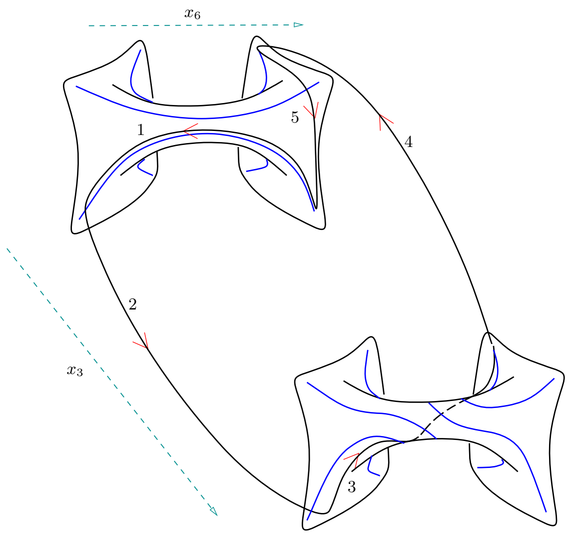

Figure 4. An example of the domain wall S_2 interpolating between the surfaces Σ_0 at $x_3 \to -\infty$ and Σ_2 at $x_3 \to +\infty$ in SU(4) SYM. S_2 is a three-dimensional manifold with two non-trivial cycles. One of them, peculiar to the domain wall, is described in the figure as a composition of five pieces. It winds twice around the M-theory circle.

study, we should consider the elements of $H_1(\widetilde{Y}/S_k, \mathbb{Z})$ and minimize their length. This would certainly require a more detailed understanding of the manifold S_k (some results can be found in [5]).

Another point to discuss is the topology. One may, in fact, suspect that the topology of the manifold S_k could have some other non-trivial cycles that could ruin our homology group computation. We think that this is not the case. The reason is as follows. If we make a large n limit keeping k/n fixed, we expect the domain wall to behave as a soliton of the effective Lagrangian of the kind

$$\mathcal{L}_{\text{eff}} \propto n^2 F(\phi, \vec{\nabla}\phi, \ldots). \tag{3.6}$$

So its tension should scale like n^2 and, most importantly, its spatial dependence and size, being determined only by $\mathcal{F}$, should be n independent. So for example, if we take a k-wall in an SU(n) gauge theory, it should have the same profile of a $k/[k, n]$-wall in a SU($n/[k, n]$) gauge theory. The only difference is in their tension whose ratio is $[k, n]^2$. In MQCD, we expect the manifold S_k of the SU(n) theory to have the same spatial properties, and in particular the same topology, of the manifold $S_{k/[k,n]}$ in SU($k/[k, n]$). The only difference is that the former winds $[k, n]$ times more around the M-theory circle. So the topology is not changed.

4. Gauge-gravity correspondence

Gauge-gravity correspondence relates a certain gauge theory to a string theory in a particular background. We now consider the gauge-gravity realization of $\mathcal{N} = 1$ SYM due to Maldacena–Nunez [6] and Klebanov–Strassler [7] solutions. The differences between these two realizations are not relevant for what we are going to say. What is important for us is the chiral symmetry breaking at the end of the cascade. We begin with a brief summary of this result, then introduce the k-wall in this framework and finally explain the emergence of $\mathbb{Z}_{[k,n]}$ strings.

The string theory under consideration is type IIB. Spacetime is composed of the 3 + 1-dimensional spacetime where the gauge theory lives, times a warped dimension corresponding to the energy scale of the gauge theory, times an internal manifold of topology $S^2 \times S^3$. n units of F_3^{RR} flux pass through the S^3 sphere. Confining strings are F1-strings and can be annihilated in units of n ending on a baryon vertex. The baryon vertex is a D3 brane wrapped on the S^3 sphere. A Chern–Simons interaction with the F_3^{RR} form requires n fundamental strings to end on the baryon vertex [8]. A domain wall is a D5 brane wrapping the S^3 sphere. The 1-strings ending on a 1-wall are now an F1-string ending on the D5 brane.

We now want to consider the k-wall that consists of k D5 branes superimposed. The phenomenon we want to see, that is, the $\mathbb{Z}_{[k,n]}$ strings bounded inside the domain wall, is a non-perturbative effect from the point of view of the domain wall effective action. The way to see it is to consider a large number of domain walls ($k, n \to \infty$ while keeping k, n fixed) and consider the 't Hooft limit of this low-energy effective action. We should thus consider the black brane description of the k D5 branes and go to the near-horizon geometry. Clearly, this is a difficult task; we do not even know the solution of the black 5-brane in these fields' background. But our goal requires much less than the full solution of the gauge-gravity dual of the k-wall effective action. We just need to know that the horizon of these k D5 branes is a three-dimensional manifold, with topology S^3 and with k units of F_3^{RR} flux passing through it. A D3 brane wrapping this manifold is a kind of baryon vertex for this 2 + 1 effective theory. The Chern–Simons interaction with the F_3^{RR} flux requires k strings to end on it.

We have thus found two baryon vertices on which strings can annihilate. One is that of the original gauge theory, living in 3 + 1 dimensions, on which n F1-strings can end. The other is the baryon vertex of the k-wall that lives on its 2 + 1 world volume. The last can annihilate k units of F1's. Strings can thus be annihilated in any integral linear combinations of n and k, and so the $\mathbb{Z}_{[k,n]}$ stability follows.

The baryon vertex living on the k-wall solves an apparent puzzle of our previous analysis. We said, in fact, that a 1-string cannot end on a k-wall (consider $[k, n] = k$ for simplicity here) but a k-string does. But what happens if we take k 1-strings? They should be able to terminate on the k-wall although the single 1-strings are not able to. What happens is explained by the k-wall baryon vertex (see figure 5). From every end point of 1-strings, a string inside the wall departs. These k-strings are then annihilated into a k-wall baryon vertex.

In the previous discussion, we have neglected the backreaction of the D3 branes on the geometry. This must be certainly taken into account for a full treatment of the problem. We can nevertheless consider the limit k fixed, $[k, n]$ fixed and $n \to \infty$. In this limit we can neglect the backreaction of the k D3 branes, and the previous discussion can be considered a valid support of our main statement.

5. Field theoretical arguments

The string inside wall phenomenon has been studied in detail in the paper [14], following the initial idea of [13]. Strings can form a bound state with a domain wall if there is a charged condensate that *does not* vanish on both sides of the wall. This basic mechanism is also the one responsible for the string–wall bound state in $\mathcal{N} = 1$ SYM. Let us consider the k-wall that interpolates between the 0 and the k vacua, where the two condensates responsible for confinement are, respectively, the monopole (0, 1) and the dyon $(k, 1)$. Strings inside walls are formed by the tunneling of the condensed particles, from one vacuum to the other. This tunneling is possible only if k and n have some divisor in common. We can, in fact, take

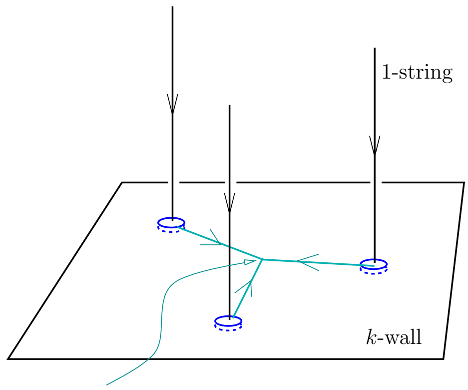

Figure 5. A baryon vertex living on the domain wall provides the mechanism for $[k, n]$ separate 1-strings to end on the wall.

a bunch of $n/[k, n]$ monopoles on one side and make them reappear on the other side as $n/[k, n]$ dyons. The charges match since we have $(0, n/[k, n])$ on the monopole side and $(n/[k, n] \cdot k, n/[k, n]) = (0, n/[k, n])$ on the dyon side. This tunneling is responsible for the formation of the $[k, n]$ confining strings inside the k-wall in SU(n) super Yang–Mills.

$\mathcal{N} = 1$ super Yang–Mills is still far from being under complete analytical control. The fact that it lies in a non-Abelian kind of confining phase, at the base of the heuristic argument, has not yet a direct and rigorous proof. To have a more solid field theoretical setting, we shall investigate in what follows the soflty broken $\mathcal{N} = 2$ and detect here the string inside wall phenomenon.

We consider the deformation from $\mathcal{N} = 2$ through a mass term $\mu \mathrm{Tr}\, \Phi^2/2$ for the adjoint chiral superfield. The theory for $\mu \ll \Lambda_{\mathcal{N}=2}$ is under analytical control, thanks to the Seiberg–Witten solution. The gauge group Abelianizes SU$(n) \to$ U$(1)^{n-1}$ and each U(1) is Higgses by the condensation of an opportune low-energy hyper-multiplet. Duality between the microscopic and macroscopic descriptions implies confinement of the original electric probe charges. This phase still persists as we change the mass μ. In particular, the n vacua are continuously deformed in the n vacua of pure $\mathcal{N} = 1$ as $\mu \gg \Lambda_{\mathcal{N}=1}$. An important difference though is that there is no Abelianization in the $\mu \to \infty$ limit and the nature of the confining phase is purely non-Abelian.

We now want to try the deformed $\mathcal{N} = 2$ technique to understand the phenomenon in which we are interested in this paper. The basic prototype of the domain wall in deformed $\mathcal{N} = 2$ has been studied in [17], at least for the simplest case $n = 2$. The SW curve for SU$(n)\mathcal{N} = 2$ SYM is

$$\begin{aligned} y^2 &= \mathcal{P}_n(z) \\ &= \tfrac{1}{4}\det(z - \phi)^2 - \Lambda^{2n}, \end{aligned} \tag{5.1}$$

where we defined for convenience the polynomial $\mathcal{P}_n$. The maximal singularity points for $\mathcal{P}_n$ are given by the solution of Douglas and Shenker [28]. There are n of these maximal singularity points. They happen when the n cuts are lined up and all the roots, a part from two of them, are doubled. One solution is when all the roots are on the real axis. The others are related by an $\mathrm{e}^{2\pi \mathrm{i}k/n}$ transformation.

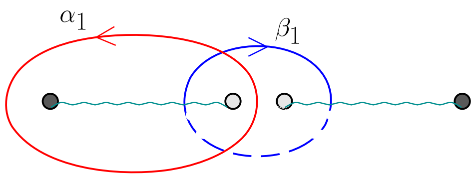

Figure 6. Roots and cycles near the real vacuum (5.2). The choice of basis is made so that α corresponds to the electric in weak coupling.

In the real case we can take $\phi = \mathrm{diag}(\phi_1, \ldots, \phi_n)$ and $\phi_j = 2\Lambda\cos\left(\pi\frac{j-1/2}{n}\right)$, and the curve is thus written in terms of the Chebyshev functions:

$$\begin{aligned}\mathcal{P}_n &= \frac{1}{4}\prod_{j=1}^{n}\left(z - 2\Lambda\cos\left(\pi\frac{j-1/2}{n}\right)\right)^2 - \Lambda^n \\ &= \left(\frac{1}{4}T_n\left(\frac{z}{2\Lambda}\right)^2 - 1\right)\Lambda^n \\ &= \left(\frac{z^2}{4} - \Lambda^2\right)U_{n-1}\left(\frac{z}{2\Lambda}\right)^2\Lambda^{n-2},\end{aligned} \tag{5.2}$$

where $U_{n-1}\left(\frac{z}{2\Lambda}\right)^2 = \prod_{j=1}^{n-1}\left(\frac{z}{2\Lambda} - \cos\frac{\pi j}{n}\right)$. For the factorization of the curve, we have used the important identity:

$$T_n^2(z) - (z^2 - 1)U_{n-1}(z) = 1. \tag{5.3}$$

We recall what the relation is between the curve and the $\mathbb{Z}_n$ strings. Massless particles appear every time there is a vanishing cycle in the SW curve. In the maximal singularity vacua there are $n-1$ double roots, and so $n-1$ massless hyper-multiplets, one for every U(1) factor in the low-energy effective action. Upon the perturbation with the mass term μ, these massless particles condensed and creates vortices (ordinary Abrikosov–Nielsen–Olesen vortices in the low-energy effective action). These $n-1$ vortices are exactly in one-to-one correspondence with the non-trivial elements of the group of confining $\mathbb{Z}_n$ strings [4, 28, 29]:

$$\begin{aligned}\mathcal{T}_k &= 4\pi\widetilde{E}_k E_k \\ &= 4\pi\sqrt{W'^2(z) + f(z)}|_{z=2\Lambda\cos(\pi k/n)} \\ &= 8\pi\mu\Lambda\sin\frac{\pi k}{n},\end{aligned} \tag{5.4}$$

where we have derived the Douglas–Shenker sine formula for the k-string tension.

For the effect we are interested in, the simplest case to consider is $n = 4$ and $k = 2$, that is, the gauge group SU(4) and the 2-wall. We choose the 2-wall to interpolate between the $h = 0$ vacuum, the real one, and the $h = 2$ vacuum, the imaginary one. Since $[k, n] = 2$, we expect the 1-string to be stable in the 2-wall background, a parallel 1-string from a bound state inside the 2-wall. The 2-string can instead terminate on the 2-wall, exactly like the 1-string terminates on the 1-wall for the gauge group SU(2).

It is better to start with the SU(2) case that is a very well-known example, and it also appears additionally as a substructure of the SU(4) problem we shall face next. Now the SW curve has degree 4 and genus 1. There is one U(1) gauge group in the low-energy theory. There are two vacua of maximal singularity: one real and one imaginary. In figures 6 and 7, we have the roots of the SW curve near the real vacuum. The first has the cycles α and

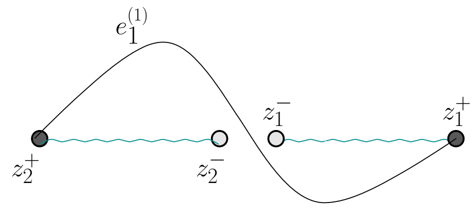

Figure 7. The same roots but now with the vanishing cycles $E^{(1)}$ corresponding to the roots that collide in the dyon vacuum.

β corresponding to the electric and magnetic components of the unbroken U(1). The second has the vanishing cycle of the particle that becomes massless in the imaginary vacuum. In the real vacuum, the particle that become massless is $E^{(0)}$ with charges (0, 1) with respect to U(1) (the first is the electric charge and the second the magnetic charges). The particle $E^{(1)}$ that is massless in the imaginary vacuum has charges $(2, -1)$. We know that the 1-string (the only string in this case) is an ANO vortex created by $E^{(0)}$. Through the screening mechanism, it can terminate on the 1-wall. The condition for this to be possible is that the flux carried by the string can be expressed as the sum of the charged particles that condense on both sides of the wall[2]:

$$\text{flux} = \nu^{(0)} e^{(0)} + \nu^{(1)} e^{(1)}. \tag{5.5}$$

This condition is solved for the 1-string carrying flux (1, 0) by

$$\nu^{(0)} = 1/2, \qquad \nu^{(1)} = -1/2. \tag{5.6}$$

We shall discuss more in detail this string–wall junction, and how to quantitatively approach the problem, in appendix.

Now let us move to the more interesting SU(4) case. The curve has eight roots and they can be divided into *plus* and *minus* roots according to the following factorization:

$$\begin{aligned}\mathcal{P}_4 &= \tfrac{1}{4}\det(z-\phi)^2 - \Lambda^8 \\ &= \left(\tfrac{1}{2}\det(z-\phi) - \Lambda^4\right)\left(\tfrac{1}{2}\det(z-\phi) + \Lambda^4\right) \\ &= \mathcal{P}_4^- \mathcal{P}_4^+.\end{aligned} \tag{5.7}$$

We then give names to the various roots:

$$z^-_{1,2,3,4}, \qquad z^+_{1,2,3,4}. \tag{5.8}$$

We want to focus our attention on the two vacua $h = 0$ and $h = 2$, respectively, the real and the imaginary vacua. In the real case, the factorization gives the following roots:

$$\mathcal{P}_4^- = \tfrac{1}{2} z^2 (z - 2\Lambda)(z + 2\Lambda) \qquad \mathcal{P}_4^+ = \tfrac{1}{2}(z - \sqrt{2}\Lambda)^2 (z + \sqrt{2}\Lambda)^2. \tag{5.9}$$

In the imaginary case,

$$\mathcal{P}_4^- = \tfrac{1}{2} z^2 (z - 2\mathrm{i}\Lambda)(z + 2\mathrm{i}\Lambda) \qquad \mathcal{P}_4^+ = \tfrac{1}{2}(z - \sqrt{2}\mathrm{i}\Lambda)^2 (z + \sqrt{2}\mathrm{i}\Lambda)^2. \tag{5.10}$$

Six of the roots are paired, and as a consequence each of the $\mathrm{U}(1)^3$ low-energy gauge groups has a massless charged hyper-multiplet. The roots shuffling is important for our purposes. To compute the charges of the massless particles in a given vacuum, we need the vanishing

[2] We use E to denote the fields and e to denote the charges and/or the cycles.

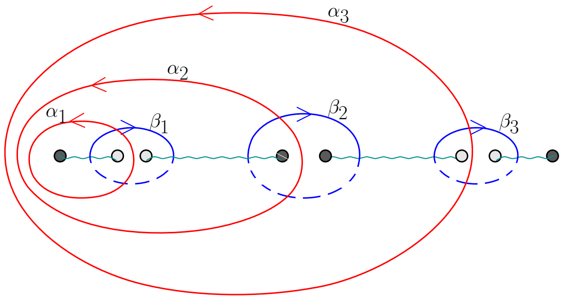

Figure 8. Roots and cycles near the real vacuum (5.2). The choice of basis is made so that the charges of massless particles are diagonal (table 1).

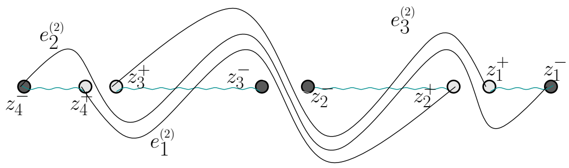

Figure 9. The same roots, but now with the vanishing cycles $e_1^{(2)}, e_2^{(2)}, e_3^{(2)}$ corresponding to the imaginary vacuum (table 2).

Table 1. Low-energy gauge groups and charged hyper-multiplets for the real vacuum (5.2), $n = 4, h = 0$.

	$\mathrm{U}(1)_1$	$\mathrm{U}(1)_2$	$\mathrm{U}(1)_3$
$e_1^{(0)}$	$(0_\mathrm{e}, 1_\mathrm{m})$		
$e_2^{(0)}$		$(0_\mathrm{e}, 1_\mathrm{m})$	
$e_3^{(0)}$			$(0_\mathrm{e}, 1_\mathrm{m})$

cycles corresponding to the given massless particles. To get them is not only necessary to know which roots collide, but also to understand the path they follow with respect to the other roots.

Passing from the vacuum $h = 0$ to the vacuum $h = 0$, the two set of roots $z^-_{1,2,3,4}$ and $z^+_{1,2,3,4}$ are shuffled independently. The roots $z^+_{1,2,3,4}$ are the one responsible for the massless particles $E_1^{(2)}, E_3^{(2)}$ and $E_1^{(0)}, E_3^{(0)}$. In this particular case, the condensation of E_1 creates the 1-string, condensation of E_2 the 2-string and condensation of E_3 the 3-string of $\mathbb{Z}_4$.

We then use the *same* basis to compute the charges for the imaginary vacuum (table 2). Of course, we could have used another basis in which the charges would have locked diagonal exactly like in the real vacuum. But since we want to study the domain wall between the two vacua, we need a description in which the three U(1)'s are expressed on the same basis in the two vacua. In figure 9, we displayed the vanishing cycles corresponding to the massless particles in the imaginary vacuum. Expanding these cycles on the basis previously given gives us the charges of table 2.

Table 2. Low-energy gauge groups and charged hyper-multiplets for the imaginary vacuum (5.2), $n = 4, h = 2$. We have used the same basis of table 1.

	$U(1)_1$	$U(1)_2$	$U(1)_3$
$e_1^{(2)}$	$(-1_e, 1_m)$	$(2_e, -1_m)$	$(-1_e, 0_m)$
$e_2^{(2)}$	$(-2_e, 1_m)$	$(2_e, -1_m)$	$(-2_e, 1_m)$
$e_3^{(2)}$	$(-1_e, 0_m)$	$(2_e, -1_m)$	$(-1_e, 1_m)$

Table 3. Low-energy gauge groups and charged hyper-multiplets for $n = 6, h = 3$.

	$U(1)_1$	$U(1)_2$	$U(1)_3$	$U(1)_4$	$U(1)_5$
e_1	$(0_e, 0_m)$	$(-1_e, 0_m)$	$(2_e, -1_m)$	$(-1_e, 1_m)$	$(0_e, 0_m)$
$e_2^{(3)}$	$(1_e, 0_m)$	$(-2_e, 1_m)$	$(2_e, -1_m)$	$(-2_e, 1_m)$	$(1_e, -1_m)$
$e_3^{(3)}$	$(2_e, -1_m)$	$(-2_e, 1_m)$	$(2_e, -1_m)$	$(-2_e, 1_m)$	$(2_e, -1_m)$
$e_4^{(3)}$	$(1_e, -1_m)$	$(-2_e, 1_m)$	$(2_e, -1_m)$	$(-2_e, 1_m)$	$(1_e, 0_m)$
$e_5^{(3)}$	$(0_e, 0_m)$	$(-1_e, 1_m)$	$(2_e, -1_m)$	$(-1_e, 0_m)$	$(0_e, 0_m)$

The screening condition (the generalization of (5.5)) now is

$$\begin{aligned} \text{flux} &= \nu_1^{(2)} e_1^{(2)} + \nu_2^{(2)} e_2^{(2)} + \nu_3^{(2)} e_3^{(2)} \\ &= \nu_1^{(0)} e_1^{(0)} + \nu_2^{(0)} e_2^{(0)} + \nu_3^{(0)} e_3^{(0)}, \end{aligned} \tag{5.11}$$

which is solved for the 2-string carrying flux $(0, 0)$, $(1, 0)$, $(0, 0)$ by

$$\begin{array}{lll} \nu_1^{(2)} = 1/2 & \nu_2^{(2)} = -1/2 & \nu_3^{(2)} = 1/2 \\ \nu_1^{(0)} = 0 & \nu_2^{(0)} = -1/2 & \nu_3^{(0)} = 0. \end{array} \tag{5.12}$$

Note that we have a redundancy, since two equations (the electric of $U(1)_1$ and $U(1)_3$) are equivalent: $0 = \alpha + 2\beta - \gamma$. This is related to the reason why the vortices created by $E_1^{(0)}$ and $E_1^{(0)}$ cannot be *screened*. Consider the 1-string with flux $(1, 0)$, $(0, 0)$, $(0, 0)$. The electric of $U(1)_1$ and $U(1)_3$) are now, respectively, $1 = \alpha + 2\beta - \gamma$ and $0 = \alpha + 2\beta - \gamma$; clearly, there is no solution to the termination condition (5.11) in this case.

We also note that the vectorial space spanned by the composite condensate $\widetilde{E}_1^{(0)} E_3^{(0)}$ is exactly the same as that spanned by $\widetilde{E}_1^{(2)} E_3^{(2)}$. The product of the fields implies the sum of the charges:

$$-e_1^{(0)} + e_3^{(0)} = -e_1^{(2)} + e_3^{(2)}. \tag{5.13}$$

So the situation is completely analogous to that described in [14]. The 1-string and the 3-string cannot terminate on the wall. If parallel to it, they are attracted to form a bound state consisting of a confining string inside the domain wall. The lowest energy configuration for the string is where the condensates reaches their minimum, and that is in the middle of the wall.

We can also analyze the SU(6) gauge theory. As before, we consider a domain wall with the $h = 0$ vacuum on on the e side. We again choose the basis of cycles so that the charges appears simple in this vacuum. There are five $U(1)_i$ gauge groups and five hyper-multiplets $E_i^{(0)}$ with diagonal charges $(0_e, 1_m)$. With $n = 6$, there are two interesting domain walls we can consider. One is the 3-wall between the 0 vacuum and the 3 vacuum. Keeping the same base choice as before, the charges of the massless particles in the 2-vacuum are given in table 3. Since $[k, n]$ is now equal to 3, we expect the 3-string to be able to terminate on the

Table 4. Low-energy gauge groups and charged hyper-multiplets for $n = 6$, $h = 2$.

	$\mathrm{U}(1)_1$	$\mathrm{U}(1)_2$	$\mathrm{U}(1)_3$	$\mathrm{U}(1)_4$	$\mathrm{U}(1)_5$
$e_1^{(2)}$	$(0_\mathrm{e}, 0_\mathrm{m})$	$(0_\mathrm{e}, 0_\mathrm{m})$	$(1_\mathrm{e}, 0_\mathrm{m})$	$(-2_\mathrm{e}, 1_\mathrm{m})$	$(1_\mathrm{e}, -1_\mathrm{m})$
$e_2^{(2)}$	$(0_\mathrm{e}, 0_\mathrm{m})$	$(-1_\mathrm{e}, 0_\mathrm{m})$	$(2_\mathrm{e}, -1_\mathrm{m})$	$(-2_\mathrm{e}, 1_\mathrm{m})$	$(2_\mathrm{e}, -1_\mathrm{m})$
$e_3^{(2)}$	$(1_\mathrm{e}, 0_\mathrm{m})$	$(-2_\mathrm{e}, 1_\mathrm{m})$	$(2_\mathrm{e}, -1_\mathrm{m})$	$(-2_\mathrm{e}, 1_\mathrm{m})$	$(1_\mathrm{e}, 0_\mathrm{m})$
$e_4^{(2)}$	$(2_\mathrm{e}, -1_\mathrm{m})$	$(-2_\mathrm{e}, 1_\mathrm{m})$	$(2_\mathrm{e}, -1_\mathrm{m})$	$(-1_\mathrm{e}, 0_\mathrm{m})$	$(0_\mathrm{e}, 0_\mathrm{m})$
$e_5^{(2)}$	$(1_\mathrm{e}, -1_\mathrm{m})$	$(-2_\mathrm{e}, 1_\mathrm{m})$	$(1_\mathrm{e}, 0_\mathrm{m})$	$(0_\mathrm{e}, 0_\mathrm{m})$	$(0_\mathrm{e}, 0_\mathrm{m})$

domain wall. The screening condition, as we can see

$$\begin{aligned} \text{3-string} &= (0,0);\,(0,0);\,(1,0);\,(0,0);\,(0,0) \\ &= \tfrac{1}{2}\big(e_3^{(3)} + e_3^{(0)} - e_2^{(3)} - e_4^{(3)} + e_1^{(3)} + e_5^{(3)}\big), \end{aligned} \tag{5.14}$$

is in fact solvable. The 1-string and 2-string are, instead, stable. The reason is the equivalence between the following equivalences of the condensates:

$$e_1^{(3)} - e_5^{(3)} = -e_2^{(0)} + e_4^{(0)}, \qquad -e_2^{(3)} + e_4^{(3)} = -e_1^{(0)} + e_5^{(0)}. \tag{5.15}$$

We can complete the discussion with the 2-wall in the SU(6) theory. Charges in the 2-vacuum are now given in table 4. The 2-string can terminate on the domain wall. The screening condition is

$$\begin{aligned} \text{2-string} &= (0,0);\,(1,0);\,(0,0);\,(0,0);\,(0,0) \\ &= \tfrac{1}{3}\big(-e_2^{(2)} + 2e_3^{(2)} - 2e_4^{(2)} + 2e_5^{(2)} - 2e_2^{(0)} - e_3^{(0)} - e_4^{(0)} - e_5^{(0)}\big). \end{aligned} \tag{5.16}$$

The 1-string is stable due to the following equality:

$$-e_1^{(2)} + e_3^{(2)} - e_5^{(2)} = e_1^{(0)} - e_3^{(0)} + e_5^{(0)}. \tag{5.17}$$

6. Peeling

The reader has probably an objection in his mind. Here, we want to promptly discuss and solve this apparent puzzle.

According to the previous statement, a 1-string cannot terminate to a 2-wall in the case where the number of colors n is even (the simplest example of $[k, n]$ non-trivial effect). But there could be a *peeling* of the 2-wall. A condensate with the required charge could thus be formed near the first peeled sheet, which corresponds to a 1-wall. The peeling seems to require a certain finite amount of energy, but this would be gained by the fact that we do not need another string at the opposite side of the wall. It appears that in this way, we gain an infinite amount of energy at the price of just a peeling of a small portion of the 2-wall.

This objection, as we expressed it, contains two mistakes.

First of all, the energy difference between a string ending on a wall and a string crossing the wall and proceeding on the other half-space is not infinite, but finite. The reason lies behind the logarithmical bending that a string produces on a wall where it terminates. When a string of tension T_S ends on a wall of tension T_W, it produces a deformation of the profile of the wall. At a large distance, this deformation is $f(r) = (T_\mathrm{S}/\pi T_\mathrm{W}) \log r$. To evaluate the energy of the wall, we perform a surface integral $T_\mathrm{W} \int 2\pi r \mathrm{d}r \sqrt{1 + f'^2}$. At a large distance, the derivative goes to zero and we can expand and we obtain two terms. The first corresponds to the energy of a flat wall. The second term is $\big(T_\mathrm{S}^2 / T_\mathrm{W}\big) \int^R \mathrm{d}r\, 1/r$. Performing the integral

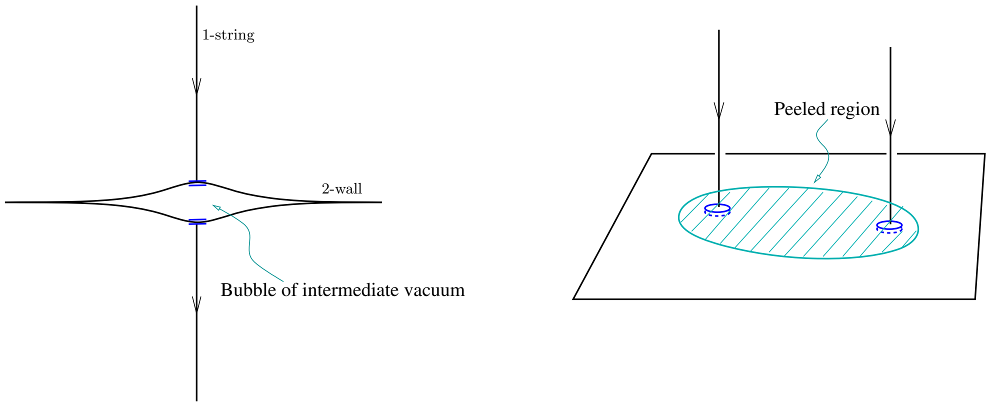

Figure 10. The correct picture for n even is the left one. When a bubble of intermediate vacuum is formed from the peeling, the total charge of the particles inside the bubble must be zero.

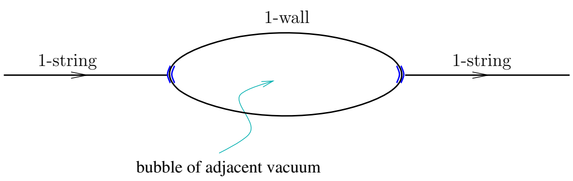

Figure 11. A 1-string broken by a 1-wall bubble.

we get $T_S f(R)$, that is, exactly the energy of a string on the other side of a flat wall. We thus see from this simple calculation that the energy of the two configurations differs only by a finite amount of energy: *the boojium*. We have thus seen that there is no infinite amount of energy in the game. A good understanding of this phenomenon can be found in extended supersymmetric theories [24, 25] where everything in BPS saturates and the boojium energy can be computed exactly as a central charge.

But there is a more important reason why the objection does not stand. When we peel the 2-wall, the bubble of intermediate vacuum must have a net zero charge. So the correct picture is the left part of figure 10. This is just because we always have to satisfy a charge conservation like (2.1). We can move apart the two strings (right part of figure 10), but there must always be a peeled region connecting the two ending points.

This could be another intuitive interpretation of the string that lies inside the wall. The string inside the wall is equivalent to a portion of the peeled wall.

This fact can be further elucidated with another example that uses only well-consolidated facts, without any relation to the string inside the wall phenomenon. The two facts we are using are as follows: (1) confining strings, in particular the 1-string, are stable objects in the vacuum and (2) a 1-string can terminate on a 1-wall. Now let us do the following experiment. We take a single 1-string in the 0-vacuum, and then create a bubble of the 1-vacuum separated from the outside by a 1-wall.

So the 1-string can be broken by the insertion of a bubble of 1-wall containing inside an adjacent vacuum (the adjacent vacuum is inside the bubble). The bubble *cannot* be broken and separated into two pieces. Otherwise we would be able to break the 1-string into two disconnected pieces, like in a quark–antiquark formation. Being the 1-string absolutely stable in this theory (there are no quark fields), we conclude that a vacuum bubble must have total charge zero inside of it. The argument just presented, made out of well-consolidated facts, gives further understanding of the peeling phenomenon and the fact that the confining string is equivalent to a portion of the peeled vacuum.

7. Domain wall effective action

The effective action of the low-energy degrees of freedom of a k-wall has been considered by Acharya–Vafa (AF) [9]. In the string theory realization of $\mathcal{N} = 1$ SYM previously considered, the k domain wall consists of k D5 branes wrapped on an S^3 sphere. In spacetime, we get a 2 + 1 brane with an $\mathcal{N} = 2$ U(k) gauge theory. The U(k) gauge theory descends directly from the gauge degrees of freedom on the branes. Here $\mathcal{N} = 2$ is in 2 + 1 dimensions, that is, four real supercharges (like $\mathcal{N} = 1$ in 3 + 1 dimensions). We know that domain walls are half-BPS saturate, and so we expect only two real supercharges corresponding to $\mathcal{N} = 1$ in 2 + 1 dimensions. The breaking of supersymmetry derives from the flux passing through the S^3 sphere. This induces an $\mathcal{N} = 1$ Chern–Simon interaction at level n. The 2 + 1 effective action, in the $\mathcal{N} = 1$ language, thus consists of a gauge U(k) multiplet with a Yang–Mills and a Chern–Simons term coupled to an adjoint superfield. Written explicitly in terms of the physical fields, it is

$$\mathcal{L}_{2+1} = \frac{1}{g^2}\mathrm{Tr}\left(-\frac{1}{2}F_{\mu\nu}F^{\mu\nu} + \mathrm{i}\chi D_\mu\gamma^\mu\chi + \mathrm{i}\psi D_\mu\gamma^\mu\psi + D_\mu\phi D^\mu\phi - \chi[\phi,\psi]\right)$$
$$+ \frac{n}{4\pi}\mathrm{Tr}\left(\frac{1}{2}\epsilon^{\mu\nu\rho}\left(A_\mu F_{\nu\rho} - \frac{2}{3}A_\mu A_\nu A_\rho\right) - \chi\chi\right) \tag{7.1}$$

The fermion λ is the supersymmetric partner of the gauge field A_μ and acquires mass through the supersymmetric Chern–Simon term. The fermion ψ is the supersymmetric partner of ϕ. Together, A_μ, λ, χ and ϕ are the field content of an $\mathcal{N} = 2$ multiplet in 2 + 1 dimensions[3]. The Chern–Simons term splits their masses, leaving only $\mathcal{N} = 1$ residual supersymmetry.

The Acharya–Vafa theory has been derived in a string theory setup. This setting is a parental to $\mathcal{N} = 1$ SYM, but not exactly the same. Many questions about the validity of the AF theory in the pure field theoretical context still remain unanswered. Every low-energy effective action on domain walls has meaning and validity up to the scale of the inverse of the wall thickness. The theory has been put to a test only for the index [9, 10] and succeeded in counting the expected number of vacua. But a more detailed understanding is certainly still needed. In particular, is also not clear how to identify the gauge degrees of freedom of the Acharya–Vafa theory with the bulk ones. Moreover, we do not even have a toy model that can reproduce the Chern–Simons term on the domain wall effective action.

With these warnings in mind, let us for the moment take the conservative approach that the Acharya-Vafa theory described part of the domain wall dynamics. We fist ask ourselves what the phase of this theory is.

First of all consider, the theory without the Chern–Simons term. It is $\mathcal{N} = 2$ in three dimensions. At a tree level, there is a moduli space given by the expectation value of the scalar field. From its holomorphic properties, it is possible to compute the non-perturbative-generated superpotential [27]. The result is that there is a run-away vacuum and $\langle\phi\rangle$ goes to infinity. The Chern–Simons term breaks $\mathcal{N} = 2$ to $\mathcal{N} = 1$ giving a topological mass to

[3] The first two lines of the Lagrangian are in fact the dimensional reduction of $\mathcal{N} = 1$ in 3 + 1, and χ, ψ are the two real components of the complex gaugino usually denoted as λ.

the photon and to one real component of the spinor. To higher loops in perturbation theory, the Chern–Simons term generates a potential for the scalar field that stabilizes the vacuum [30]. From the fact that the domain wall is BPS [1], it is believed that the theory has a stable supersymmetric vacuum at a certain value of $\langle\phi\rangle = 0$. This is consistent with what is known from the central charge of the bulk theory that determines the tension of the walls.

Now consider the coupling dependence versus n; the coupling scales like $1/g^2 \propto n/\Lambda$, where Λ is the dynamical scale of the original $\mathcal{N} = 1$ theory in four dimensions. There are two ways of performing a large n limit. One is to send n to infinity while keeping k/n fixed. This, in fact, would be the proper large k limit for the effective theory on the domain wall. Another way is to send n to infinity while keeping k fixed. This is the case we shall explore now. It is simpler, since the theory on the domain wall maintains as fixed the gauge group size and decreases the coupling.

For simplicity we shall thus restrict ourselves to a 2-wall, that is $k = 2$, and take n large. This is the simplest and most tractable example to consider. The reason is that perturbation theory is an expansion in powers of $g^2/\mu \propto \Lambda/(n\mu)$, where μ is the energy scale. The theory becomes strongly coupled at energies of order Λ/n. The Chern–Simons term generates a topological mass for the photon of order $m_{\mathrm{CS}} = ng^2/(4\pi) \propto \Lambda$ and thus where the theory is weakly coupled. From this, we can infer the following conclusion. For arbitrarily large n and fixed k, the theory is weakly coupled at all scales. The large n analysis (with k fixed) is thus under perturbative control. The theory is in the topological massive phase since the Chern–Simons mass acts at a scale where the gauge coupling is still small. More detailed analysis seems to confirm that this is the case for all n and k [15].

This seems to go in the opposite direction of our claim. In the Acharya–Vafa theory, there is no sign of confinement and strings inside the domain wall. To understand this point, and why this is not really in contradiction with our claim, we need to open a brief parenthesis on a supersymmetric toy model discussed in [14] which shares similar characteristics.

As already introduced in section 5, the purpose of Auzzi *et al* [14] is to understand, and find explicit realization of, the mechanism that creates confining strings inside domain walls. One of the two models presented in [14] is a supersymmetric theory that has important similarities with that discussed in the present paper.

The supersymmetric theory considered in [14] is $\mathcal{N} = 2$ gauge theory, with the gauge group $\mathrm{U}(2) = \mathrm{SU}(2) \times \mathrm{U}(1)/Z_2$, with no matter hyper-multiplets. The following superpotential which breaks the extended supersymmetry down to $\mathcal{N} = 1$ is then added:

$$W = \alpha \mathrm{Tr}\left(\frac{\Phi^3}{3} - \xi\Phi\right). \tag{7.2}$$

Classically, we have three vacua, with ϕ equal to

$$\begin{pmatrix} \sqrt{\xi} & 0 \\ 0 & \sqrt{\xi} \end{pmatrix}, \qquad \begin{pmatrix} \sqrt{\xi} & 0 \\ 0 & -\sqrt{\xi} \end{pmatrix}, \qquad \begin{pmatrix} -\sqrt{\xi} & 0 \\ 0 & -\sqrt{\xi} \end{pmatrix}. \tag{7.3}$$

The first and the last vacua preserve the non-Abelian SU(2) gauge symmetry. Strong coupling effects à la Seiberg and Witten will then split each of them into two vacua (the monopole and dyon vacua). The vacuum in the middle preserves only the $\mathrm{U}(1) \times \mathrm{U}(1)$ gauge symmetry, and is not split. We, thus, expect in total five vacua, for generic values of ξ. The five vacua are depicted in figure 12. The value of u_2 is ξ for all five vacua. It is not modified by quantum corrections. The Coulomb vacuum in the middle is not modified by quantum correction either.

In the limit $\sqrt{\xi} \gg \Lambda$, the Coulomb vacuum is such that the electric coupling is small. As ξ decreases and becomes of order Λ, the Coulomb vacuum enters a strong coupling regime. At the critical value $\xi = \Lambda^2$, the Coulomb vacuum lies exactly in the monopole singularity

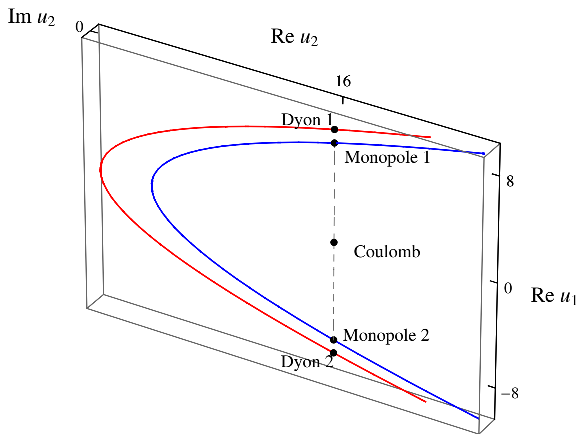

Figure 12. Five vacua of the model ($\Lambda = 1, \xi = 4$). The dashed line corresponds to the composite domain wall.

and coalesces with two monopole vacua. Around this critical value, the Coulomb vacuum is such that the magnetic coupling is small, so we can use the same set of low-energy effective variables to describe both the Coulomb and the confining vacua.

The value of the superpotential in the two confining vacua, monopole-1 and monopole-2, is

$$W = \pm\tfrac{4}{3}\alpha(\xi - \Lambda^2)^{3/2}. \tag{7.4}$$

In the Coulomb vacuum, the superpotential vanishes. Hence, the BPS bound for the tension of the wall interpolating between monopole-1 and monopole-2 vacua, if it existed, would be twice that of the BPS wall interpolating between the Coulomb and confining vacua. The latter walls will be referred to as elementary. The former wall can be called composite. The three values of the superpotential are allineated in the complex plane.

In this theoretical setup, no BPS wall interpolating between two confining vacua, monopole-1 and monopole-2, exists. In other words, a composite wall built of two elementary walls at a finite distance from each other does not exist. Supersymmetric solutions correspond to solutions of a dynamical system determined by the first-order equations, starting from monopole-1, following the profile W and ending in monopole-2. A field configuration interpolating between monopole-1 and monopole-2 is always time dependent; it represents two elementary walls moving under the influence of a repulsive force between them (see [32]). This force falls off exponentially with the wall separation. Alternatively we can say that the composite wall exists and is BPS, but the distance between the elementary ones is infinite.

In order to avoid this problem and stabilize the composite domain wall, an extra term is introduced in the superpotential:

$$W = \alpha\left(\mathrm{Tr}\left(\frac{\Phi^3}{3} - \xi\Phi\right) + \frac{\mathrm{i}\mu}{2}(\mathrm{Tr}\,\Phi)^2\right), \tag{7.5}$$

where ξ and μ are real mass parameters. The values of the superpotential in both confining vacua change due to the double trace operator of the same quantity. The Coulomb vacuum is instead unaffected. The net result is that the three values of the superpotential are no more allineated in the complex plane.

The tension of the BPS domain wall is given by the absolute value of the difference of the superpotentials at two vacua between which the given wall interpolates. For this reason, if the composite BPS walls exist at $\mu \neq 0$, the composite wall will be stable. Direct numerical investigation has revealed that the composite domain wall is indeed BPS, and the distance between the two elementary walls is stabilized by the μ deformation.

We would like to understand the localization of the (massive) gauge field on the wall as a quasimodulus σ localized on the wall world volume. The condition that typical energies in the low-energy theory must be $\ll 1/d$ cannot be met then. In this formulation, it makes no sense to speak of localization and reduction to 2 + 1 dimensions. The wall at $\sigma = \pi$ corresponds, in fact, to the infinite distance between the two elementary ones.

Thus, although the low-energy description in the Seiberg–Witten-motivated model at hand is not of the sine-Gordon type, the quasimodulus-based low-energy description is still valid at $|\sigma| \ll \pi$: a mass term $m\sigma^2$ is generated.

The conclusion is that (1) the present model is a realization of the string-inside-wall phenomenon and (2) this string cannot be interpreted as a domain line in a 2 + 1 effective action.

In the previous example, confinement inside domain walls is not a phenomenon that can be, in general, captured by a 2 + 1 low-energy effective action. This is always the case where the model under consideration shares a fundamental property with the example just described. The key properties are as follows: (i) the domain wall interpolates between two vacua where a common particle condenses and creates a vortex and (ii) there is an intermediate vacuum, a *true* one, where the particle is massive. In this case the confinement cannot be understood from a 2 + 1 effective action, of whatsoever kind.

The strings inside walls in $\mathcal{N} = 1$ SYM fall exactly in this category. Thus, the fact that Acharya-Vafa theory is in the topological massive phases and shows no sign of confining strings, this should not be viewed as a contradiction to our statement.

To understand better this last remark, let us refer to the example given in section 5, in particular the 2-wall in the SU(4) gauge theory. The composite operator $\widetilde{E}_1^{(0)} E_3^{(0)}$ in the 0-vacuum is exactly the same as the composite operator $\widetilde{E}_1^{(2)} E_3^{(2)}$ in the 2-vacuum. This is the scalar field responsible for the creation of the domain line living inside the domain wall. This is obtained by the winding of the relative phase between the operator on the left side and on the right side (what we called σ in [14]), This phase *cannot* be implemented on a 2 + 1 effective action. This would be possible only if every configuration with constant σ could be obtained as an adiabatic deformation of the lowest state ($\sigma = 0$, the basic domain wall) while preserving the condition of validity of the effective action, that is energy $\ll 1/d$ with d the thickness of the wall. The configuration $\sigma = \pi$ is the one in which the condensates of $\widetilde{E}_1^{(0)} E_3^{(0)}$ and $\widetilde{E}_1^{(2)} E_3^{(2)}$ vanish in the middle of the composite wall. This can be obtained only when the two elementary walls are at infinite distance. Since $d \to \infty$ as $\sigma \to \pi$, the condition of validity of the effective action inevitably vanishes. a low-energy effective action can only see the fluctuation of σ around the zero and is inevitably blind to the topological structure which is essential to explain the formation of the string-wall bound state.

8. Conclusion

We now conclude, summarizing the main result of this paper. We have argued that in the presence of a k-wall of SU(n) super Yang–Mills, there are $\mathbb{Z}_{[k,n]}$ topologically stable strings. According to the direction in which we orient the string with respect to the wall we can have different scenarios. If the strings are perpendicular, they cross the domain wall and continue to the other half-space. They can change their n-ality only modulo $[k, n]$. If the strings are parallel to the wall, they will feel an attractive force toward it. Their more energetically favorable position will be inside the wall, as a bound state.

We have used various arguments to support our claim: one heuristic, one from MQCD, another from the gauge-gravity correspondence and finally a field theoretical one.

Still it is not completely clear what the nature of the low-energy effective action on the domain walls is. Is not clear if the Acharya-Vafa theory is the right one or a more generic, maybe non-local, description is needed. The low-energy effective action would certainly be a usefull tool in the approach to these problems. But we should also keep in mind that the string inside wall phenomenon is not expected to be detectable in any 2 + 1 effective action [14]. Although localized on the wall, it is still a fully 3 + 1 effect[4].

The string-wall junction is not yet under quantitative control. The most crude approximation we can take has been described in section 5. Due to the non-locality of the fields, we are unable to make a global ansatz for the fields and study the junction in a solitonic, weakly coupled approach.

Acknowledgments

I am grateful to R Auzzi, A Yung, and M Shifman. for various discussions and the collaboration in [14]. I am very grateful to J Evslin for usefull discussion in Bruxelles in May 2007. I want to thank D Tong for discussions at the Isaac Newton Institute in September 2007. Part of this work has been done in the Summer of 2007 while I was supported by the Marie Curie grant MEXT-CT-2004-013510. My work is now supported by DOE grant DE-FG02-94ER40823.

Appendix A. The string-wall junction

At the base of the present paper is the fact that a confining 1-string can end on a 1-wall in $\mathcal{N} = 1$ SYM. The heuristic interpretation is that the flux carried by the string is screened by any possible combination of condensates at the two sides of the domain wall. If such a combination exists then we say that the string can terminate on the domain. Otherwise we have the string inside walls phenomenon described in the paper.

The purpose of this Appendix is to describe in more detail the string-wall junction and the bi-condensate effect that is responsible for the screening of the string flux. To do this we work in the context of $\mathcal{N} = 2$ broken by a small μ mass term. The two vacua are a monopole $(0, 1)$ and a dyon $(2, -1)$, that respectively condense and give confinement.

We describe the domain wall between the monopole ((0) vacuum) and the dyon ((1) vacuum) much on the same lines as has been done in [17]. We neglect completely the boundary effects, and consider the condensate to drop to zero like a step function. There is an inner layer between both the condensates and the energy density is non-vanishing. We set equal to 1 this non-zero energy density. Stabilization of the wall is given introducing a scalar field u which has value -1 on the (0) side and $+1$ on the (1) side. The tension as a function of the distance d of the domain wall is $d + 2/d$. Stabilization is given by the balance of forces, given by the derivatives $-1 + 2/d^2 = 0$. 1 is the force, per unit of area, due to the energy density on the internal layer; it is constant and always inward (negative). The other term is a positive force (outward) equal to the derivative of the scalar field u. The result is a stable domain wall with tension $T_{\mathrm{W}} = 2\sqrt{2}$ and thickness $d = \sqrt{2}$.

In a generic setup, where the shape is not necessary that of a straight domain wall, the situation changes slightly, and essential now is the use of the stabilization of forces argument. The energy density always gives an inward force of magnitude one, both on the (0) and (1)

[4] The previous version of the paper contained an erroneous interpretation of the Acharya-Vafa theory in relation to the string inside wall phenomenon.

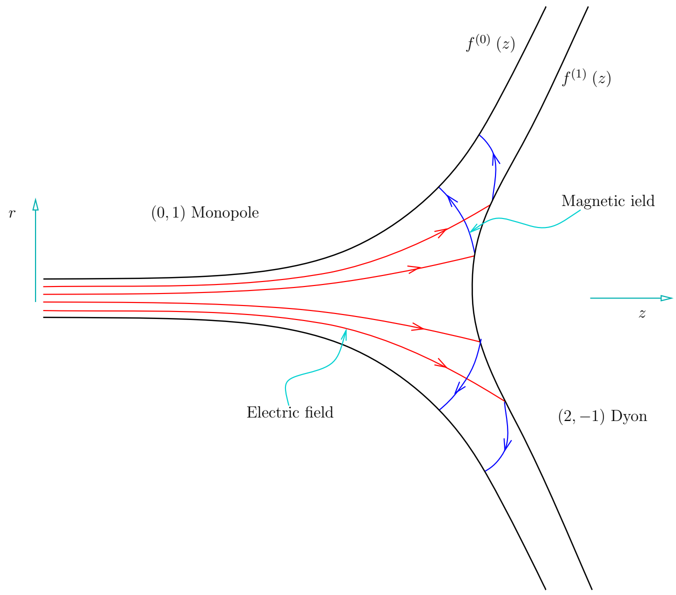

Figure A1. The string-wall junction in our basic approximation.

sides. For the scalar field we have to solve the Laplacian equation with Dirichlet boundary conditions on the two sides:

$$\triangle u = 0, \qquad u|_{(0)} = -1, \qquad u|_{(1)} = +1 \tag{A.1}$$

The force from the scalar field is given by the gradient $\vec{\nabla} u$ and directed outward.

A similar schematization can be used for the confining string. We consider the string in the monopole vacuum, and thus it will carry electric flux equal to 2π. The tension of this vortex is the sum of the electric energy contribution, $E^2/2$ times the area, plus the vacuum energy contribution, 1 times the area. In total it is $2\pi/r^2 + \pi r^2$. The force per unit of area acting on the monopole surface is $(1/2\pi r)$ times the derivative ∂_r of the previous expression: $-2/r^4+1 = 0$. The balance of forces is thus $E^2/2 - 1 = 0$. Again, there is an inward force of magnitude 1 due to the energy density. The other force is $E^2/2$ and is directed outward, and this is because the electric field is acting on the monopole current that creates it. The result is a vortex with tension $T_S = 2\sqrt{2}\pi$ and radius $r = \sqrt[4]{2}$.

We are now ready to combine the two objects, vortex and wall, together in the junction of figure A1. We use cylindrical coordinates r, z, θ. The vortex comes from the monopole side and is continuously merged with the domain wall. In this crude approximation, the solution of the junction is given by the two profiles $r = f^{(0)}(z)$ and $r = f^{(1)}(r)$, respectively, the edge of the monopole condensate and the edge of the dyon condensate. In the middle there are no condensates and the energy density acts as an inward force on every surface element of the monopole and dyon borders.

To obtain the other forces acting on the surfaces we have to solve the Laplace equation for three functions: $u, \varphi^{\mathrm{e}}, \varphi^{\mathrm{m}}$. The first is the scalar field already introduced before. The second and the third are, respectively, the electric and magnetic potentials: $\vec{E} = \vec{\nabla}\varphi^{\mathrm{e}}$, $\vec{B} = \vec{\nabla}\varphi^{\mathrm{m}}$. Now the essential ingredients are the boundary conditions. On the monopole side the electric field must be tangential to the surface $f^{(0)}$ (Neumann boundary condition) while the magnetic field must be perpendicular (Dirichlet boundary conditions):

$$\vec{n}\cdot\vec{\nabla}\varphi^{\mathrm{e}}|_{(0)}=0, \qquad \varphi^{\mathrm{m}}|_{(0)}=\mathrm{const}=0 \tag{A.2}$$

The electric field is generated by monopole current, like in the solitonic vortex. The force that the electric field exerts on the surface is *outward* with magnitude $E^2/2$. The magnetic field is generated by the monopole condensate itself, like in a solitonic Q-ball. To obtain the boundary condition on the dyon side we have to perform th correct duality transformation. The combination $2\varphi^{\mathrm{e}}-\varphi^{\mathrm{m}}$ is the one with Dirichlet boundary conditions, while $\varphi^{\mathrm{e}}+2\varphi^{\mathrm{m}}$ is the one with Neumann boundary condition:

$$\vec{n}\cdot\vec{\nabla}(\varphi^{\mathrm{e}}+2\varphi^{\mathrm{m}})|_{(1)}=0, \qquad (2\varphi^{\mathrm{e}}-\varphi^{\mathrm{m}})|_{(1)}=\mathrm{const}=0 \tag{A.3}$$

Finally, we writhe the force balance condition on the two surfaces. On the monopole side we have that the force that the electric field exerts on the surface is *outward* with magnitude $E^2/2$. The force is again $B^2/2$, but this time directed *inward*.

$$\left.\left(-1+\vec{\nabla}u+\frac{\vec{E}^2}{2}-\frac{\vec{B}^2}{2}\right)\right|_{(0)}=0 \tag{A.4}$$

On the dyon surface we have a very similar equation but with the correct duality transformation:

$$\left.\left(-1+\vec{\nabla}u+\frac{(-\vec{E}+2\vec{B})^2}{2}-\frac{(2\vec{E}+\vec{B})^2}{2}\right)\right|_{(1)}=0 \tag{A.5}$$

Up to now we have provided the equations that govern the string-wall junction. Solving them is not easy, it could in principle be done with a recursive numerical technic similar to the one used in [33]. We start with certain functions $f^{(0)}$ and $f^{(1)}$. Then we solve the Laplace equations with the given boundary conditions. We thus have the forces that act on the two surfaces. We then modify the functions in order to minimize the forces and reach zero at the end of the iteration. Although easy to say, it is a numerically challenging problem.

We have given the equations that determine the junction profiles $f^{(0)}$ and $f^{(1)}$, and we have given an implementable method to solve the equation numerically. The string-wall junction, from this solitonic point of view, is composed by a vortex, two Q-balls (the bi-condensate that screens the vortex flux) and a domain wall. As a further step is necessary to understand how to connect the fields, even in the bulk where the monopole and the dyon condense. Non-locality is the main obstacle to this understanding. It is, in fact, clear that a global solution with a gauge potential φ, $\vec{A}$ is not possible.

References

[1] Dvali G R and Shifman M A 1997 Domain walls in strongly coupled theories *Phys. Lett.* B **396** 64
Dvali G R and Shifman M A 1997 Domain walls in strongly coupled theories *Phys. Lett.* B **407** 452 (erratum) (arXiv:hep-th/9612128)

[2] Witten E 1997 Branes and the dynamics of QCD *Nucl. Phys.* B **507** 658 (arXiv:hep-th/9706109)

[3] Bolognesi S 2005 Domain walls and flux tubes *Nucl. Phys.* B **730** 127 (arXiv:hep-th/0507273)

[4] Hanany A, Strassler M J and Zaffaroni A 1998 Confinement and strings in MQCD *Nucl. Phys.* B **513** 87 (arXiv:hep-th/9707244)

[5] Volovich A 1999 Domain walls in MQCD and Monge-ampere equation *Phys. Rev.* D **59** 065005 (arXiv:hep-th/9801166)

[6] Maldacena J M and Nunez C 2001 Towards the large N limit of pure N = 1 super Yang Mills *Phys. Rev. Lett.* **86** 588 (arXiv:hep-th/0008001)

[7] Klebanov I R and Strassler M J 2000 Supergravity and a confining gauge theory: Duality cascades and chiSB-resolution of naked singularities *J. High Energy Phys.* JHEP08(2000)052 (arXiv:hep-th/0007191)

[8] Witten E 1998 Baryons and branes in anti de Sitter space *J. High Energy Phys.* JHEP07(1998)006 (arXiv:hep-th/9805112)
[9] Acharya B S and Vafa C 2001 On domain walls of N = 1 supersymmetric Yang–Mills in four dimensions (arXiv:hep-th/0103011)
[10] Ritz A, Shifman M and Vainshtein A 2002 Counting domain walls in N = 1 super Yang–Mills *Phys. Rev.* D **66** 065015 (arXiv:hep-th/0205083)
[11] Deser S, Jackiw R and Templeton S 1982 Topologically massive gauge theories *Ann. Phys.* **140** 372
Deser S, Jackiw R and Templeton S 1988 Topologically massive gauge theories *Ann. Phys.* **185** (406.1988 APNYA,281,409 (1988 APNYA,281,409-449.2000)) (erratum)
[12] Polyakov A M 1977 Quark confinement and topology of Gauge groups *Nucl. Phys.* B **120** 429
[13] Dvali G, Nielsen H B and Tetradis N 2007 Localization of Gauge fields and monopole tunnelling (arXiv:0710.5051 [hep-th])
[14] Auzzi R, Bolognesi S, Shifman M and Yung A 2008 Confinement and localization on domain walls (arXiv:0807.1908 [hep-th])
[15] Witten E 1999 Supersymmetric index of three-dimensional gauge theory (arXiv:hep-th/9903005)
[16] Campos A, Holland K and Wiese U J 1998 Complete wetting in supersymmetric QCD or why QCD strings can end on domain walls *Phys. Rev. Lett.* **81** 2420 (arXiv:hep-th/9805086)
[17] Kaplunovsky V S, Sonnenschein J and Yankielowicz S 1999 Domain walls in supersymmetric Yang–Mills theories *Nucl. Phys.* B **552** 209 (arXiv:hep-th/9811195)
[18] Dvali G R, Gabadadze G and Kakushadze Z 1999 BPS domain walls in large N supersymmetric QCD *Nucl. Phys.* B **562** 158 (arXiv:hep-th/9901032)
[19] Gabadadze G and Shifman M A 2000 D-walls and junctions in supersymmetric gluodynamics in the large N limit suggest the existence of heavy hadrons *Phys. Rev.* D **61** 075014 (arXiv:hep-th/9910050)
[20] 't Hooft G 1978 On the phase transition towards permanent quark confinement *Nucl. Phys.* B **138** 1
[21] Donagi R and Witten E 1996 Supersymmetric Yang–Mills Theory And Integrable Systems *Nucl. Phys.* B **460** 299 (arXiv:hep-th/9510101)
[22] Shifman M and Yung A 2003 Domain walls and flux tubes in N = 2 SQCD: D-brane prototypes *Phys. Rev.* D **67** 125007 (arXiv:hep-th/0212293)
[23] Isozumi Y, Nitta M, Ohashi K and Sakai N 2005 All exact solutions of a 1/4 Bogomol'nyi-prasad-Sommerfield equation *Phys. Rev.* D **71** 065018 (arXiv:hep-th/0405129)
[24] Sakai N and Tong D 2005 Monopoles, vortices, domain walls and D branes: The rules of interaction *J. High Energy Phys.* JHEP03(2005)019 (arXiv:hep-th/0501207)
[25] Auzzi R, Shifman M and Yung A 2005 Studying boojums in N = 2 theory with walls and vortices *Phys. Rev.* D **72** 025002 (arXiv:hep-th/0504148)
[26] Auzzi R, Shifman M and Yung A 2006 Domain lines as fractional strings *Phys. Rev.* D **74** 045007 (arXiv:hep-th/0606060)
[27] Affleck I, Harvey J A and Witten E 1982 Instantons and (super)symmetry breaking In (2+1)-dimensions *Nucl. Phys.* B **206** 413
[28] Douglas M R and Shenker S H 1995 Dynamics of SU(N) supersymmetric gauge theory *Nucl. Phys.* B **447** 271 (arXiv:hep-th/9503163)
[29] Bolognesi S 2005 The holomorphic tension of vortices *J. High Energy Phys.* JHEP01(2005)044 (arXiv:hep-th/0411075)
[30] Armoni A and Hollowood T J 2005 Sitting on the domain walls of N = 1 super Yang–Mills *J. High Energy Phys.* JHEP07(2005)043 (arXiv:hep-th/0505213)
[31] Armoni A and Hollowood T J 2006 Interactions of domain walls of SUSY Yang–Mills as D-branes *J. High Energy Phys.* JHEP02(2006)072 (arXiv:hep-th/0601150)
[32] Portugues R and Townsend P K 2002 Intersoliton forces in the Wess-Zumino model *Phys. Lett.* B **530** 227 (arXiv:hep-th/0112077)
[33] Bolognesi S and Gudnason S B 2006 Soliton junctions in the large magnetic flux limit *Nucl. Phys..* B **754** 293 (arXiv:hep-th/0606065)
[34] Greensite J 2003 The confinement problem in lattice gauge theory *Prog. Part. Nucl. Phys.* **51** 1 (arXiv:hep-lat/0301023)
[35] Armoni A and Shifman M 2003 Remarks on stable and quasi-stable k-strings at large N *Nucl. Phys.* B **671** 67 (arXiv:hep-th/0307020)
[36] Lucini B, Teper M and Wenger U 2004 Glueballs and k-strings in SU(N) gauge theories: Calculations with improved operators *J. High Energy Phys.* JHEP06(2004)012 (arXiv:hep-lat/0404008)
[37] Bolognesi S 2005 Large N, Z(N) strings and bag models *Nucl. Phys.* B **730** 150 (arXiv:hep-th/0507286)